

Monomial ideals of graphs with loops

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Abstract

We investigate, using the notion of linear quotients, significative classes of connected graphs whose monomial edge ideals, not necessarily squarefree, have linear resolution, in order to compute standard algebraic invariants of the polynomial ring related to these graphs modulo such ideals. Moreover we are able to determine the structure of the ideals of vertex covers for the edge ideals associated to the previous classes of graphs which can have loops on any vertex. Lastly, it is shown that these ideals are of linear type.

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Introduction

In [8] it was first showed how some algebraic properties related to remarkable classes of graphs hold when appropriate loops are put.

In this article we are interested in studying standard algebraic properties of monomial ideals arising from the edges of some graphs, the so-called edge ideals, and exposing situations for which such properties are preserved when we add loops to them. The generators of ideals of vertex covers for the edge ideals associated to graphs with loops are also examined, proving that these ideals are of linear type.

Let \mathcal{G} be a graph on n vertices v_1, \dots, v_n . An algebraic object attached to \mathcal{G} is the edge ideal $I(\mathcal{G})$, a monomial ideal of the polynomial ring in n variables

$R = K[X_1, \dots, X_n]$, K a field.

When \mathcal{G} is a loopless graph, $I(\mathcal{G})$ is generated by squarefree monomials of degree two in R , $I(\mathcal{G}) = (\{X_i X_j \mid \{v_i, v_j\} \text{ is an edge of } \mathcal{G}\})$, but if \mathcal{G} is a graph having loops $\{v_i, v_i\}$, among the generators of $I(\mathcal{G})$ there are also non-squarefree monomials X_i^2 , $i = 1, \dots, n$.

For a relevant class of connected graphs with loops we prove that their edge ideals have linear resolution, by using the technique of studying the linear quotients of such ideals, as previously employed in [9].

More precisely, it is examined a wide class of squarefree edge ideals associated to the connected graphs \mathcal{H} , on n vertices, consisting of the union of a complete graph K_m , $m < n$, and star graphs with centers the vertices of K_m . By adding loops on some vertices of K_m , we introduce a new class of non-squarefree edge ideals associated to the connected graphs $\mathcal{K} = \mathcal{H} \cup \{v_j, v_j\}$, for some $j = 1, \dots, m$. We prove that $I(\mathcal{H})$ and $I(\mathcal{K})$ have linear resolution. We also give formulae for standard invariants of $R/I(\mathcal{H})$ and $R/I(\mathcal{K})$ such as dimension, projective dimension, depth, Castelnuovo-Mumford regularity. In [7] the computation of such invariants is made for the symmetric algebras of subclasses of squarefree edge ideals generated by s -sequences associated to \mathcal{G} .

Some algebraic aspects linked to the minimal vertex covers for such classes of graphs can be considered. Keeping in mind the one to one correspondence between minimal vertex covers of any graph and minimal prime ideals of its edge ideal, we generalize to a graph with loops the notion of ideal of (minimal) vertex covers and determine the structure of the ideals of vertex covers $I_c(\mathcal{H})$ and $I_c(\mathcal{K}')$ for the classes of edge ideals associated to \mathcal{H} and $\mathcal{K}' = \mathcal{H} \cup \{v_i, v_i\}$, for some $i = 1, \dots, n$. We may observe that \mathcal{K}' is larger than \mathcal{K} because loops on \mathcal{K}' stay also on vertices that don't belong to K_m . Moreover we prove that the symmetric and the Rees algebras of $I_c(\mathcal{H})$ and $I_c(\mathcal{K}')$ over R are isomorphic, namely such ideals of linear type.

The work is subdivided as follows. In section 1 the classes of graphs \mathcal{H}, \mathcal{K} and their edge ideals are analyzed. In particular, starting from \mathcal{H} , we introduce \mathcal{K} and consider its edge ideal whose ordered generators are $X_1 X_{m+1}, X_1 X_{m+2}, \dots, X_1 X_{m+i_1}, X_1 X_2, X_2 X_{m+i_1+1}, \dots, X_2 X_{m+i_1+i_2}, X_2 X_3, X_1 X_3, X_3 X_{m+i_1+i_2+1}, \dots, X_3 X_{m+i_1+i_2+i_3}, X_3 X_4, X_2 X_4, X_1 X_4, \dots, X_m X_{m+i_1+\dots+i_{m-1}+1}, \dots, X_m X_{m+i_1+\dots+i_m}, X_{t_1}^2, \dots, X_{t_l}^2$, where $\{t_1, \dots, t_l\} \subseteq \{1, \dots, m\}$ and $n = m + i_1 + \dots + i_m$.

According to results that characterize monomial ideals with linear quotients ([2, 4]), it is showed that the edge ideals of \mathcal{H} and \mathcal{K} have linear resolution. As an application, standard algebraic invariants are calculated.

In section 2 we examine the structure of the ideals of vertex covers for the classes of edge ideals associated to \mathcal{H} and \mathcal{K}' using the description of the ideals of vertex covers for the edge ideals associated to the complete graphs

and the star graphs that make them up.

We remark that the ideal of vertex covers of $I(\mathcal{K}')$, when \mathcal{K}' has loops on all its n vertices, has the unique generator $X_1 \cdots X_m \cdots X_n$.

In section 3 we investigate the Rees algebra of $I_c(\mathcal{H})$ and $I_c(\mathcal{K}')$. We recall that if $I = (f_1, \dots, f_s)$ is an ideal of R , the Rees algebra $\mathfrak{R}(I)$ of I is defined to be the R -graded algebra $\bigoplus_{i \geq 0} I^i$. Let $\varphi : R[T_1, \dots, T_s] \rightarrow \mathfrak{R}(I) = R[f_1 t, \dots, f_s t]$ be an epimorphism of graded R -algebras defined by $\varphi(T_i) = f_i t$, $i = 1, \dots, s$, and $N = \ker \varphi$ be the ideal of presentation of $\mathfrak{R}(I)$. If N is generated by linear relations, namely R -homogeneous elements of degree 1, then I is said of linear type. Several classes of ideals of R of linear type are known. For instance, ideals generated by d -sequences and M -sequences are of linear type ([6, 10, 1]). We show that the ideals of vertex covers $I_c(\mathcal{H})$ and $I_c(\mathcal{K}')$ are of linear type.

1 Linear resolutions and invariants

Let $R = K[X_1, \dots, X_n]$ be the polynomial ring in n variables over a field K with $\deg X_i = 1$, for all $i = 1, \dots, n$. For a monomial ideal $I \subset R$ we denote by $G(I)$ its unique minimal set of monomial generators.

Definition 1.1. A *vertex cover* of a monomial ideal $I \subset R$ is a subset C of $\{X_1, \dots, X_n\}$ such that each $u \in G(I)$ is divided by some $X_i \in C$.

The vertex cover C is called *minimal* if no proper subset of C is a vertex cover of I .

Let $h(I)$ denote the minimal cardinality of the vertex covers of I .

Definition 1.2. A monomial ideal $I \subset R$ is said to have *linear quotients* if there is an ordering u_1, \dots, u_t of monomials belonging to $G(I)$ with $\deg(u_1) \leq \dots \leq \deg(u_t)$ such that the colon ideal $(u_1, \dots, u_{j-1}) : (u_j)$ is generated by a subset of $\{X_1, \dots, X_n\}$, for $2 \leq j \leq t$.

Remark 1.1. A monomial ideal I of R generated in one degree that has linear quotients admits a linear resolution ([2], Lemma 4.1).

For a monomial ideal I of R having linear quotients with respect to the ordering u_1, \dots, u_t of the monomials of $G(I)$, let $q_j(I)$ denote the number of the variables which is required to generate the ideal $(u_1, \dots, u_{j-1}) : (u_j)$, and set $q(I) = \max_{2 \leq j \leq t} q_j(I)$.

Remark 1.2. The integer $q(I)$ is independent on the choice of the ordering of the generators that gives linear quotients ([5]).

In this section we study classes of monomial ideals generated in degree two arising from graphs that have linear quotients, that is equivalent to say that they have linear resolution ([4]).

Let's introduce some preliminary notions.

Let \mathcal{G} be a graph and $V(\mathcal{G}) = \{v_1, \dots, v_n\}$ be the set of its vertices. We put $E(\mathcal{G}) = \{\{v_i, v_j\} | v_i \neq v_j, v_i, v_j \in V(\mathcal{G})\}$ the set of edges of \mathcal{G} and $L(\mathcal{G}) = \{\{v_i, v_i\} | v_i \in V(\mathcal{G})\}$ the set of loops of \mathcal{G} . Hence $\{v_i, v_j\}$ is an edge joining v_i to v_j and $\{v_i, v_i\}$ is a loop of the vertex v_i . Set $W(\mathcal{G}) = E(\mathcal{G}) \cup L(\mathcal{G})$. If $L(\mathcal{G}) = \emptyset$, the graph \mathcal{G} is said simple or loopless, otherwise \mathcal{G} is a graph with loops.

A graph \mathcal{G} on n vertices v_1, \dots, v_n is *complete* if there exists an edge for all pairs $\{v_i, v_j\}$ of vertices of \mathcal{G} . It is denoted by K_n .

If $V(\mathcal{G}) = \{v_1, \dots, v_n\}$ and $R = K[X_1, \dots, X_n]$ is the polynomial ring over a field K such that each variable X_i corresponds to the vertex v_i , the *edge ideal* $I(\mathcal{G})$ associated to \mathcal{G} is the ideal $(\{X_i X_j | \{v_i, v_j\} \in W(\mathcal{G})\}) \subset R$.

Note that the non-zero edge ideals are those generated by monomials of degree 2. This implies that $I(\mathcal{G})$ is a graded ideal of S of initial degree 2, that is $I(\mathcal{G}) = \oplus_{i \geq 2} (I(\mathcal{G}))_i$. If $W(\mathcal{G}) = \emptyset$, then $I(\mathcal{G}) = (0)$.

First it is examined a relevant wide class of squarefree edge ideals associated to connected graphs \mathcal{H} , on n vertices, consisting of the union of a complete graph K_m , $m < n$, and star graphs in the vertices of K_m .

More precisely, $\mathcal{H} = K_m \cup \text{star}_j(k)$, where K_m is the complete graph on m vertices v_1, \dots, v_m , $m < n$, and $\text{star}_j(k)$ is the star graph on k vertices with center v_j , for some $j = 1, \dots, m$, $k \leq n - m$. One has:

$$\begin{aligned} I(\mathcal{H}) = & (X_1 X_{m+1}, X_1 X_{m+2}, \dots, X_1 X_{m+i_1}, X_1 X_2, X_2 X_{m+i_1+1}, X_2 X_{m+i_1+2}, \dots, \\ & X_2 X_{m+i_1+i_2}, X_2 X_3, X_1 X_3, X_3 X_{m+i_1+i_2+1}, X_3 X_{m+i_1+i_2+2}, \dots, X_3 X_{m+i_1+i_2+i_3}, \\ & X_3 X_4, X_2 X_4, X_1 X_4, \dots, X_m X_{m+i_1+i_2+\dots+i_{m-1}+1}, X_m X_{m+i_1+i_2+\dots+i_{m-1}+2}, \dots, \\ & X_m X_{m+i_1+i_2+\dots+i_{m-1}+i_m}) \subset R = K[X_1, \dots, X_n]. \\ |G(I(\mathcal{H}))| = & i_1 + i_2 + \dots + i_m + \frac{m(m-1)}{2}. \end{aligned}$$

Proposition 1.1. $I(\mathcal{H})$ has a linear resolution.

Proof. $I(\mathcal{H})$ is the edge ideal of the graph \mathcal{H} with $n = m + i_1 + i_2 + \dots + i_m$ vertices and $\ell = i_1 + i_2 + \dots + i_m + \frac{m(m-1)}{2} = n - m + \frac{m(m-1)}{2} = n + \frac{m(m-3)}{2}$ edges. Set f_1, \dots, f_ℓ the squarefree monomial generators of $I(\mathcal{H})$. It results:

$$\begin{aligned} (f_1) : (f_2) &= (X_{m+1}), \\ (f_1, f_2) : (f_3) &= (X_{m+1}, X_{m+2}), \\ &\dots, \\ (f_1, \dots, f_{i_1-1}) : (f_{i_1}) &= (X_{m+1}, X_{m+2}, \dots, X_{m+i_1-1}), \\ (f_1, \dots, f_{i_1}) : (f_{i_1+1}) &= (X_{m+1}, X_{m+2}, \dots, X_{m+i_1}), \end{aligned}$$

$(f_1, \dots, f_{i_1+1}) : (f_{i_1+2}) = (X_1 X_{m+1}, X_1 X_{m+2}, \dots, X_1 X_{m+i_1}, X_1) = (X_1),$
 $(f_1, \dots, f_{i_1+2}) : (f_{i_1+3}) = (X_1, X_{m+i_1+1}),$
 $\dots\dots\dots,$
 $(f_1, \dots, f_{i_1+i_2+1}) : (f_{i_1+i_2+2}) = (X_1, X_{m+i_1+1}, \dots, X_{m+i_1+i_2}),$
 $(f_1, \dots, f_{i_1+i_2+2}) : (f_{i_1+i_2+3}) = (X_{m+1}, X_{m+2}, \dots, X_{m+i_1}, X_2),$
 $(f_1, \dots, f_{i_1+i_2+3}) : (f_{i_1+i_2+4}) = (X_2, X_1),$
 $\dots\dots\dots,$
 $(f_1, \dots, f_{i_1+i_2+i_3+2}) : (f_{i_1+i_2+i_3+3}) = (X_2, X_1, X_{m+i_1+i_2+1}, \dots, X_{m+i_1+i_2+i_3}),$
 and so on up to
 $(f_1, \dots, f_{\ell-1}) : (f_{\ell}) = (X_1, \dots, X_{m-1}, X_{m+i_1+i_2+\dots+i_{m-1}+1}, \dots, X_{m+i_1+i_2+\dots+i_m-1}).$
 Hence $I(\mathcal{H})$ has linear quotients. According to results in [4] about monomial ideals with linear quotients, it follows that the edge ideal $I(\mathcal{H})$ has a linear resolution. \square

Remark 1.3. R. Fröberg has proved that the edge ideal of a simple graph has a linear resolution if and only if its complementary graph is chordal ([4], Theorem 9.2.3). It is possible to prove that the edge ideal $I(\mathcal{H})$ of Proposition 1.1 has a linear resolution using such characterization.

The study of the linear quotients is useful in order to investigate algebraic invariants of $R/I(\mathcal{H})$: the dimension, $\dim_R(R/I(\mathcal{H}))$, the depth, $\text{depth}(R/I(\mathcal{H}))$, the projective dimension, $\text{pd}_R(R/I(\mathcal{H}))$ and the Castelnuovo-Mumford regularity, $\text{reg}_R(R/I(\mathcal{H}))$.

Lemma 1.1. *Let $R = K[X_1, \dots, X_n]$ and $I(\mathcal{H}) \subset R$. Then:*

$$h(I(\mathcal{H})) = m.$$

Proof. The minimal cardinality of the vertex covers of $I(\mathcal{H})$ is $h(I(\mathcal{H})) = m$, being $C = \{X_1, \dots, X_m\}$ a minimal vertex cover of $I(\mathcal{H})$ by construction. \square

Lemma 1.2. *Let $R = K[X_1, \dots, X_n]$ and $I(\mathcal{H}) \subset R$. Then:*

$$q(I(\mathcal{H})) = m + \max_{1 \leq j \leq m} i_j - 2.$$

Proof. By the computation of the linear quotients (Proposition 1.1), the maximum number of the variables which is required to generate the ideal $(f_1, \dots, f_{h-1}) : (f_h)$, for $h = 1, \dots, \ell$, is given by $(m-1) + \max_{1 \leq j \leq m} i_j - 1$. It follows that $q(I_q(\mathcal{H})) = m + \max_{1 \leq j \leq m} i_j - 2$. \square

Theorem 1.1. *Let $R = K[X_1, \dots, X_n]$ and $I(\mathcal{H}) \subset R$. Then:*

- 1) $\dim_R(R/I(\mathcal{H})) = n - m$.
- 2) $\text{pd}_R(R/I(\mathcal{H})) = m + \max_{1 \leq j \leq m} i_j - 1$.

3) $\text{depth}_R(R/I(\mathcal{H})) = n - m - \max_{1 \leq j \leq m} i_j + 1$.

4) $\text{reg}_R(R/I(\mathcal{H})) = 1$.

Proof. 1) One has $\dim_R(R/I(\mathcal{H})) = \dim_R R - h(I(\mathcal{H}))$ ([3]). Hence $\dim_R(R/I(\mathcal{H})) = n - m$, by Lemma 1.1.

2) The length of the minimal free resolution of $R/I(\mathcal{H})$ over R is equal to $q(I(\mathcal{H})) + 1$ ([5], Corollary 1.6). Then $\text{pd}_R(R/I(\mathcal{H})) = m + \max_{1 \leq j \leq m} i_j - 1$.

3) As a consequence of 2), by Auslander-Buchsbaum formula, one has $\text{depth}_R(R/I(\mathcal{H})) = n - \text{pd}_R(R/I(\mathcal{H})) = n - m - \max_{1 \leq j \leq m} i_j + 1$.

4) $I(\mathcal{H})$ has a linear resolution, then $\text{reg}_R(R/I(\mathcal{H})) = 1$. \square

Starting from the class of edge ideals associated to \mathcal{H} and adding loops on some vertices of K_m , we now analyze a larger class of non-squarefree edge ideals associated to connected graphs $\mathcal{K} = \mathcal{H} \cup \{v_j, v_j\}$, for some $j = 1, \dots, m$. Let \mathcal{K} be the connected graph with n vertices v_1, \dots, v_n and K_m , $m < n$, be the complete subgraph of \mathcal{K} with vertices v_1, \dots, v_m , such that

$$I(\mathcal{K}) = (X_1X_{m+1}, X_1X_{m+2}, \dots, X_1X_{m+i_1}, X_1X_2, X_2X_{m+i_1+1}, X_2X_{m+i_1+2}, \dots, X_2X_{m+i_1+i_2}, X_2X_3, X_1X_3, X_3X_{m+i_1+i_2+1}, X_3X_{m+i_1+i_2+2}, \dots, X_3X_{m+i_1+i_2+i_3}, X_3X_4, X_2X_4, X_1X_4, \dots, X_mX_{m+i_1+i_2+\dots+i_{m-1}+1}, X_mX_{m+i_1+i_2+\dots+i_{m-1}+2}, \dots, X_mX_{m+i_1+i_2+\dots+i_{m-1}+i_m}, X_{t_1}^2, \dots, X_{t_l}^2) \subset R = K[X_1, \dots, X_n], \text{ with } \{t_1, \dots, t_l\} \subseteq \{1, \dots, m\} \text{ and } n = m + i_1 + \dots + i_m.$$

$$|G(I(\mathcal{K}))| = n - (m - l) + \frac{m(m-1)}{2}.$$

Taking in account the notion of linear quotients, it is proved that the edge ideal $I(\mathcal{K})$ has still a linear resolution.

Proposition 1.2. $I(\mathcal{K})$ has a linear resolution.

Proof. $I(\mathcal{K})$ is the edge ideal of the graph \mathcal{K} with $n = m + i_1 + i_2 + \dots + i_m$ vertices, $\ell = i_1 + i_2 + \dots + i_m + \frac{m(m-1)}{2} = n - m + \frac{m(m-1)}{2} = n + \frac{m(m-3)}{2}$ edges and $l \leq m$ loops $\{v_{t_1}, v_{t_1}\}, \dots, \{v_{t_l}, v_{t_l}\}$. Set $f_1, \dots, f_\ell, f_{\ell+1}, \dots, f_{\ell+l}$ the monomial generators of $I(\mathcal{K})$, of which the last l are not squarefree.

The ideals $(f_1, \dots, f_{h-1}) : (f_h)$, for $h = 1, \dots, \ell$, have been computed in Proposition 1.1. Moreover, it results:

$$(f_1, \dots, f_\ell) : (f_{\ell+1}) = (X_2, \dots, X_m, X_{m+1}, \dots, X_{m+i_1}), \text{ if } v_1 \in \mathcal{K} \text{ has a loop,}$$

$$(f_1, \dots, f_\ell) : (f_{\ell+1}) = (X_1, \dots, X_{t_1-1}, X_{t_1+1}, \dots, X_m, X_{m+i_1+\dots+i_{t_1-1}+1}, \dots, X_{m+i_1+\dots+i_{t_1-1}+i_{t_1}}), \text{ if } \{v_{t_1}, v_{t_1}\} \neq \{v_1, v_1\};$$

.....,

$$(f_1, \dots, f_{\ell+l-1}) : (f_{\ell+l}) = (X_1, \dots, X_{t_l-1}, X_{t_l+1}, \dots, X_m, X_{m+i_1+\dots+i_{t_l-1}+1}, \dots, X_{m+i_1+\dots+i_{t_l-1}+i_{t_l}}), \text{ if } \{v_{t_l}, v_{t_l}\} \neq \{v_m, v_m\},$$

$$(f_1, \dots, f_{\ell+l-1}) : (f_{\ell+l}) = (X_1, \dots, X_{m-1}, X_{m+i_1+\dots+i_{m-1}+1}, \dots, X_{m+i_1+\dots+i_m}), \text{ if } v_m \in \mathcal{K} \text{ has a loop.}$$

Hence $I(\mathcal{K})$ has linear quotients, that is equivalent to say that $I(\mathcal{K})$ has a linear resolution ([4]). \square

The study of the linear quotients as in Proposition 1.2 is useful in order to investigate algebraic invariants of $R/I(\mathcal{K})$.

Lemma 1.3. *Let $R = K[X_1, \dots, X_n]$ and $I(\mathcal{K}) \subset R$. Then:*

$$h(I(\mathcal{K})) = m.$$

Proof. The minimal cardinality of the vertex covers of $I(\mathcal{K})$ is $h(I(\mathcal{K})) = m$, being $C = \{X_1, \dots, X_m\}$ a minimal vertex cover of $I(\mathcal{K})$ by construction. \square

Lemma 1.4. *Let $R = K[X_1, \dots, X_n]$ and $I(\mathcal{K}) \subset R$. Then:*

$$q(I(\mathcal{K})) = m + \max_{t_1 \leq j \leq t_l} i_j - 1, \quad l \leq m.$$

Proof. By the computation of the linear quotients (Proposition 1.2), the maximum number of the variables which is required to generate the ideal $(f_1, \dots, f_{k-1}) : (f_k)$, for $k = 1, \dots, \ell + l$, $l \leq m$, is given by $(m - 1) + \max_{t_1 \leq j \leq t_l} i_j$. It follows that $q(I_q(\mathcal{K})) = m + \max_{t_1 \leq j \leq t_l} i_j - 1$. \square

Theorem 1.2. *Let $R = K[X_1, \dots, X_n]$ and $I(\mathcal{K}) \subset R$. Then:*

- 1) $\dim_R(R/I(\mathcal{K})) = n - m$.
- 2) $\text{pd}_R(R/I(\mathcal{K})) = m + \max_{t_1 \leq j \leq t_l} i_j$, $l \leq m$.
- 3) $\text{depth}_R(R/I(\mathcal{K})) = n - m - \max_{t_1 \leq j \leq t_l} i_j$, $l \leq m$.
- 4) $\text{reg}_R(R/I(\mathcal{K})) = 1$.

Proof. 1) One has $\dim_R(R/I(\mathcal{K})) = \dim_R R - h(I(\mathcal{K}))$ ([3]). Hence $\dim_R(R/I(\mathcal{K})) = n - m$, by Lemma 1.3.

2) The length of the minimal free resolution of $R/I(\mathcal{K})$ over R is equal to $q(I(\mathcal{K})) + 1$ ([5], Corollary 1.6). Then $\text{pd}_R(R/I(\mathcal{K})) = m + \max_{t_1 \leq j \leq t_l} i_j$, $l \leq m$.

3) As a consequence of 2), by Auslander-Buchsbaum formula, one has $\text{depth}_R(R/I(\mathcal{K})) = n - \text{pd}_R(R/I(\mathcal{K})) = n - m - \max_{t_1 \leq j \leq t_l} i_j$, $l \leq m$.

4) $I(\mathcal{K})$ has a linear resolution, then $\text{reg}_R(R/I(\mathcal{K})) = 1$. \square

Remark 1.4. When the graph \mathcal{K} has at least a loop on a vertex that don't belong to K_m , then its edge ideal has not linear quotients. In fact, it can be verified there is no ordering of the monomials $f_1, \dots, f_s \in G(I(\mathcal{K}))$, $s = n - (m - l) + \frac{m(m-1)}{2}$, such that $(f_1, \dots, f_{j-1}) : (f_j)$ is generated by a subset of $\{X_1, \dots, X_n\}$, for $2 \leq j \leq s$.

2 Ideals of vertex covers

Definition 2.1. Let \mathcal{G} be a graph with vertex set $V(\mathcal{G}) = \{v_1, \dots, v_n\}$. A subset C of $V(\mathcal{G})$ is said a *minimal vertex cover* for \mathcal{G} if:

- (1) every edge of \mathcal{G} is incident with one vertex in C ;
- (2) there is no proper subset of C with this property.

If C satisfies condition (1) only, then C is called a *vertex cover* of \mathcal{G} and C is said to cover all the edges of \mathcal{G} .

The smallest number of vertices in any minimal vertex cover of \mathcal{G} is said *vertex covering number*. We denote it by $\alpha_0(\mathcal{G})$.

We consider some algebraic aspects linked to the minimal vertex covers of a graph \mathcal{G} with set of edges $E(\mathcal{G})$ and set of loops $L(\mathcal{G})$.

Let $I(\mathcal{G}) = (\{X_i X_j \mid \{v_i, v_j\} \in W(\mathcal{G}) = E(\mathcal{G}) \cup L(\mathcal{G})\})$ be the edge ideal associated to \mathcal{G} . When $\{v_i, v_i\}$ is a loop of \mathcal{G} and v_i belongs to a vertex cover of the graph, such a loop can be thought to have a double covering that preserves the minimality.

There exists a one to one correspondence between the minimal vertex covers of \mathcal{G} and the minimal prime ideals of $I(\mathcal{G})$.

In fact, \wp is a minimal prime ideal of $I(\mathcal{G})$ if and only if $\wp = (C)$, for some minimal vertex cover C of \mathcal{G} ([11], Prop. 6.1.16). Hence $\text{ht } I(\mathcal{G}) = \alpha_0(\mathcal{G})$.

Thus the primary decomposition of the edge ideal of \mathcal{G} is given by $I(\mathcal{G}) = (C_1) \cap \dots \cap (C_p)$, where C_1, \dots, C_p are the minimal vertex covers of \mathcal{G} .

Definition 2.2. Let $I \subset R = K[X_1, \dots, X_n]$ be a monomial ideal. The *ideal of (minimal) covers* of I , denoted by I_c , is the ideal of R generated by all monomials $X_{i_1} \cdots X_{i_k}$ such that $(X_{i_1}, \dots, X_{i_k})$ is an associated minimal prime ideal of I .

If $I(\mathcal{G})$ is the edge ideal of a graph \mathcal{G} , we call $I_c(\mathcal{G})$ the *ideal of vertex covers* of $I(\mathcal{G})$.

Property 2.1.
$$I_c(\mathcal{G}) = \left(\bigcap_{\substack{\{v_i, v_j\} \in E(\mathcal{G}) \\ i \neq j}} (X_i, X_j) \right) \cap (X_k \mid \{v_k, v_k\} \in L(\mathcal{G}), k \neq i, j).$$

We want to examine the structure of the ideals of vertex covers for the class of squarefree edge ideals associated to \mathcal{H} . In the sequel, we rename $v_{\alpha_1}, \dots, v_{\alpha_m}$ the vertices of the complete graph K_m , $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_m = n$, and $\alpha_1 = 1$ if there is not the star graph with center α_1 .

The following two lemmas give explicitly the generators of the ideals of vertex covers for the edge ideals of a complete graph and a star graph, respectively. Their proofs are an easy application of Property 2.1.

Lemma 2.1. *The ideal of vertex covers of $I(K_m)$ is generated by m monomials and it is $I_c(K_m) = (X_2X_3 \cdots X_m, X_1X_3 \cdots X_m, \dots, X_1X_2 \cdots X_{m-1})$.*

Lemma 2.2. *The ideal of vertex covers of $I(\text{star}_j(k))$ is generated by two monomials, namely $X_1 \cdots X_{j-1}X_{j+1} \cdots X_{k-1}, X_j$.*

Let's study the structure of the ideal of vertex covers of $I(\mathcal{H})$ when at least a vertex of K_m has degree $m - 1$.

Proposition 2.1. *Let \mathcal{H} be the connected graph with n vertices v_1, \dots, v_n formed by the union of a complete graph $K_m, m < n$, with vertices $v_{\alpha_1}, \dots, v_{\alpha_m}, 1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_m = n$, and of a star graph $\text{star}_{\alpha_1}(\alpha_1)$ on vertices v_1, \dots, v_{α_1} , or a star graph $\text{star}_{\alpha_i}(\alpha_i - \alpha_{i-1})$ with center v_{α_i} , for some $i = 2, \dots, m$. The ideal of vertex covers $I_c(\mathcal{H})$ has m monomial squarefree generators and it is:*

*$(X_1 \cdots X_{\alpha_1-1}X_{\alpha_2} \cdots X_{\alpha_m}, X_{\alpha_1}X_{\alpha_3} \cdots X_{\alpha_m}, \dots, X_{\alpha_1} \cdots X_{\alpha_{m-1}})$, if $i = 1$,
 $(X_{\alpha_2} \cdots X_{\alpha_m}, \dots, X_{\alpha_1} \cdots X_{\alpha_{i-1}}X_{\alpha_{i-1}+1} \cdots X_{\alpha_{i+1}-1}X_{\alpha_{i+1}} \cdots X_{\alpha_m}, \dots, X_{\alpha_1} \cdots X_{\alpha_{m-1}})$,
otherwise.*

Proof. Because the one to one correspondence between minimal vertex covers of a graph and minimal prime ideals of its edge ideal, for the complete graph

K_m it follows that $I(K_m) = \bigcap_{i=1}^m \mathcal{P}_{\alpha_i}$, where $\mathcal{P}_{\alpha_i} = (X_{\alpha_1}, \dots, X_{\alpha_{i-1}}, X_{\alpha_{i+1}}, \dots, X_{\alpha_m})$. The vertex covers of the graph \mathcal{H} will be:

$m-1$ vertex covers of K_m , but $\{v_{\alpha_1}, \dots, v_{\alpha_{i-1}}, v_{\alpha_{i+1}}, \dots, v_{\alpha_m}\}$, by Lemma 2.1; two vertex covers related to $\text{star}_{\alpha_i}(\alpha_i - \alpha_{i-1})$, $\{v_{\alpha_1}, \dots, v_{\alpha_{i-1}}, v_{\alpha_{i-1}+1}, \dots, v_{\alpha_{i+1}-1}, v_{\alpha_{i+1}}, \dots, v_{\alpha_m}\}$, $\{v_{\alpha_1}, \dots, v_{\alpha_m}\}$, for some $i \neq 1$, as a consequence of Lemma 2.2.

For the last ones, let $\mathcal{P}_{\overline{\alpha_i}} = (X_{\alpha_1}, \dots, X_{\alpha_{i-1}}, X_{\alpha_{i-1}+1}, \dots, X_{\alpha_{i+1}-1}, X_{\alpha_{i+1}}, \dots, X_{\alpha_m})$, $\mathcal{P} = (X_{\alpha_1}, \dots, X_{\alpha_m})$, respectively, be the associated minimal prime ideals but $\mathcal{P} \supseteq \mathcal{P}_{\alpha_j}, j \neq i$. So $I(\mathcal{H}) = \mathcal{P}_{\alpha_1} \cap \dots \cap \mathcal{P}_{\alpha_{i-1}} \cap \mathcal{P}_{\overline{\alpha_i}} \cap \mathcal{P}_{\alpha_{i+1}} \cap \dots \cap \mathcal{P}_{\alpha_m}$. On the other hand, it is clear what $\mathcal{P}_{\overline{\alpha_1}}$ denotes. Hence the thesis follows. \square

Remark 2.1. Proposition 2.1 can be generalized by considering two or more star graphs with centers as many vertices of K_m . It is sufficient to iterate that procedure for each pair of vertex covers related to the star graphs which are present.

Finally, let's consider the structure of the ideal of vertex covers of $I(\mathcal{H})$ when all the vertices of K_m have degree at least m .

Theorem 2.1. *Let \mathcal{H} be the connected graph with n vertices v_1, \dots, v_n formed by the union of a complete graph $K_m, m < n$, with vertices $v_{\alpha_1}, \dots, v_{\alpha_m}$,*

$1 < \alpha_1 < \alpha_2 < \dots < \alpha_m = n$, and of m star graphs with centers each of the vertices of K_m . The ideal of vertex covers of $I(\mathcal{H})$ has $m + 1$ monomial squarefree generators and it is

$$I_c(\mathcal{H}) = (X_{\alpha_1} X_{\alpha_2} \cdots X_{\alpha_m}, X_1 \cdots X_{\alpha_1-1} X_{\alpha_2} \cdots X_{\alpha_m}, X_{\alpha_1} X_{\alpha_1+1} \cdots X_{\alpha_2-1} X_{\alpha_3} \cdots X_{\alpha_m}, \dots, X_{\alpha_1} \cdots X_{\alpha_{m-2}} X_{\alpha_{m-2}+1} \cdots X_{\alpha_{m-1}-1} X_{\alpha_m}, X_{\alpha_1} \cdots X_{\alpha_{m-1}} X_{\alpha_{m-1}+1} \cdots X_{\alpha_m-1}).$$

Proof. A minimal vertex cover of \mathcal{H} must be $\{v_{\alpha_1}, \dots, v_{\alpha_m}\}$ and there cannot exist minimal vertex covers with a smaller number of vertices. So $\alpha_0(\mathcal{H}) = m$. Other minimal vertex covers of \mathcal{H} are those related to star graphs on α_1 and $\alpha_i - \alpha_{i-1}$ vertices, $\text{star}_{\alpha_1}(\alpha_1)$ and $\text{star}_{\alpha_i}(\alpha_i - \alpha_{i-1})$ respectively, for any $i = 2, \dots, m$, in which the centers are missing. With the notations of Proposition 2.1, let $\mathcal{P} = (X_{\alpha_1}, \dots, X_{\alpha_m})$, $\mathcal{P}_{\overline{\alpha_1}} = (X_1, \dots, X_{\alpha_1-1}, X_{\alpha_2}, \dots, X_{\alpha_m})$, $\mathcal{P}_{\overline{\alpha_i}} = (X_{\alpha_1}, \dots, X_{\alpha_{i-1}}, X_{\alpha_{i-1}+1}, \dots, X_{\alpha_i-1}, X_{\alpha_{i+1}}, \dots, X_{\alpha_m})$, for any $i = 2, \dots, m$, be the associated minimal prime ideals. Then $I(\mathcal{H}) = \mathcal{P} \cap \left(\bigcap_{i=1}^m \mathcal{P}_{\overline{\alpha_i}} \right)$, and $\text{ht } I(\mathcal{H}) = m$. Thus $I_c(\mathcal{H}) = (X_{\alpha_1} \cdots X_{\alpha_m}, X_1 \cdots X_{\alpha_1-1} X_{\alpha_2} \cdots X_{\alpha_m}, \dots, X_{\alpha_1} \cdots X_{\alpha_{m-1}} X_{\alpha_{m-1}+1} \cdots X_{\alpha_m-1})$, and the thesis follows. \square

Example 2.1. Let \mathcal{H} be the connected graph with $V(\mathcal{H}) = \{v_1, \dots, v_4, \dots, v_{11}\}$ given by $K_4 \cup \text{star}_1(1) \cup \text{star}_2(3) \cup \text{star}_3(1) \cup \text{star}_4(2)$. The ideal of vertex covers is

$$I_c(\mathcal{H}) = (X_1 X_2 X_3 X_4, X_2 X_3 X_4 X_5, X_1 X_3 X_4 X_6 X_7 X_8, X_1 X_2 X_4 X_9, X_1 X_2 X_3 X_{10} X_{11}).$$

Let's now analyze the structure of the ideals of vertex covers for the class of non-squarefree edge ideals associated to the connected graphs on n vertices $\mathcal{K}' = \mathcal{H} \cup \{v_i, v_i\}$, for some $i = 1, \dots, n$. Note that the class \mathcal{K}' is larger than \mathcal{K} because \mathcal{K}' may have loops on vertices that don't belong to K_m .

First let's enunciate two lemmas that give the generators of the ideals of vertex covers for the edge ideals of a complete graph with loops and a star graph with loops, respectively.

Their proofs are a direct consequence of Property 2.1.

Lemma 2.3. Let K'_m be the complete graph with loops having vertices v_1, \dots, v_m . The ideal of vertex covers of $I(K'_m)$ is generated at most by $m-1$ monomials.

In particular,

- (a) if there are loops in all the vertices, $I_c(K'_m) = (X_1 X_2 \cdots X_m)$,
- (b) if there are loops in $r < m$ vertices, v_{t_1}, \dots, v_{t_r} , $\{t_1, \dots, t_r\} \subseteq \{1, \dots, m\}$, $I_c(K'_m)$ has $m-r$ generators and it is $(\{X_{\sigma_1} \cdots X_{\sigma_{m-1}} \mid \sigma_j = t_j, \forall j = 1, \dots, r; \sigma_i \in \{1, \dots, m\} \setminus \{t_1, \dots, t_r\}, \forall i \neq j\})$.

Lemma 2.4. Let $star'_n(n)$ be the star graph with loops having vertices v_1, \dots, v_n . The ideal of vertex covers of $I(star'_n(n))$ has at most 2 generators. In particular,
(a) if there is a unique loop in the center v_n , $I_c(star'_n(n)) = (X_n)$,
(b) if the loops are in the vertices v_1, \dots, v_{n-1} , $I_c(star'_n(n)) = (X_1 \cdots X_{n-1})$,
(c) if there are loops in the center and other vertices v_{t_1}, \dots, v_{t_s} , $\{t_1, \dots, t_s\} \subseteq \{1, \dots, n-1\}$, $I_c(star'_n(n)) = (X_{t_1} \cdots X_{t_s} X_n)$.

The structure of the ideal of vertex covers of $I(K')$ is well described by the following

Theorem 2.2. Let K' be the connected graph with n vertices v_1, \dots, v_n formed by the union of: (i) the complete graph K_m , $m < n$, with vertices $v_{\alpha_1}, \dots, v_{\alpha_m}$, $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_m = n$; (ii) star graphs $star_{\alpha_i}(\alpha_i - \alpha_{i-1})$ on vertices $v_{\alpha_{i-1}+1}, \dots, v_{\alpha_i}$, $\forall i = 1, \dots, m$, and the index α_0 indicates 0; (iii) loops in some vertices, say $v_{\alpha_2}, v_{\alpha_4}, v_{\alpha_5}, v_{\alpha_{m-3}}, v_{\alpha_{m-1}}, v_{\alpha_{i-1}+j_1}, \dots, v_{\alpha_{i-1}+j_{p_i}}$, where $\{j_1, \dots, j_{p_i}\} \subseteq \{1, \dots, \alpha_i - \alpha_{i-1} - 1\}$. The ideal of vertex covers of $I(K')$ has at most $m + 1$ monomial generators and it is

$$I_c(K') = (X_{j_1} \cdots X_{j_{p_1}} X_{\alpha_1} X_{\alpha_1+j_1} \cdots X_{\alpha_1+j_{p_1}} X_{\alpha_2} X_{\alpha_2+j_1} \cdots X_{\alpha_{m-1}+j_{p_m}} X_{\alpha_m}, \\ X_1 \cdots X_{\alpha_1-1} X_{\alpha_1+j_1} \cdots X_{\alpha_1+j_{p_1}} X_{\alpha_2} X_{\alpha_2+j_1} \cdots X_{\alpha_{m-1}+j_{p_m}} X_{\alpha_m}, \\ X_{j_1} \cdots X_{j_{p_1}} X_{\alpha_1} X_{\alpha_1+1} \cdots X_{\alpha_2-1} X_{\alpha_2} X_{\alpha_2+j_1} \cdots X_{\alpha_{m-1}+j_{p_m}} X_{\alpha_m}, \dots, \\ X_{j_1} \cdots X_{\alpha_{m-3}+j_{p_{m-2}}} X_{\alpha_{m-2}} X_{\alpha_{m-2}+1} \cdots X_{\alpha_{m-1}-1} X_{\alpha_{m-1}} X_{\alpha_{m-1}+j_1} \cdots \\ X_{\alpha_{m-1}+j_{p_m}} X_{\alpha_m}, X_{j_1} \cdots X_{\alpha_{m-2}+j_{p_{m-1}}} X_{\alpha_{m-1}} X_{\alpha_{m-1}+1} \cdots X_{\alpha_m-1}).$$

Proof. The correspondence between minimal vertex covers of the graph \mathcal{H} and minimal prime ideals of the edge ideal of \mathcal{H} extends to the graph K' . So the number of minimal prime ideals \wp of $I(K')$ is the same than that of minimal primes of $I(\mathcal{H})$. Moreover, by Property 2.1, the generators of the prime ideals \wp are those related to \mathcal{H} multiplied by monomials X_i whenever $X_i \notin \wp$, $i = 1, \dots, n$. The latter monomials represent the vertices with loops of K' for which the corresponding variables are missing in \wp . The assertion holds through some computation taking in consideration Theorem 2.1. \square

Significative particular cases concern the graphs of the class such that:
(a) some star graphs are missing, (b) the loops lie only on the vertices of K_m , and (c) the loops lie only on the vertices not belonging to K_m .

Corollary 2.1. Let K' be as in Theorem 2.2, but there are in it less than m star graphs $star_{\alpha_i}(\alpha_i - \alpha_{i-1})$, suppose for $i = 2, 3, 6, m-3, m-2, m$. The ideal of vertex covers of $I(K')$ has at most m monomial generators and it is
 $I_c(K') = (X_{\alpha_1+j_1} \cdots X_{\alpha_1+j_{p_1}} X_{\alpha_2} X_{\alpha_2+j_1} \cdots X_{\alpha_2+j_{p_2}} X_{\alpha_3} X_{\alpha_4} \cdots X_{\alpha_{m-1}+j_{p_m}} X_{\alpha_m}, \\ X_{\alpha_1} X_{\alpha_1+1} \cdots X_{\alpha_2-1} X_{\alpha_2} X_{\alpha_2+j_1} \cdots X_{\alpha_2+j_{p_2}} X_{\alpha_3} X_{\alpha_4} \cdots X_{\alpha_{m-1}+j_{p_m}} X_{\alpha_m}, \dots, \\ X_{\alpha_1} \cdots X_{\alpha_{m-3}+j_{p_{m-2}}} X_{\alpha_{m-2}} X_{\alpha_{m-1}} X_{\alpha_{m-1}+j_1} \cdots X_{\alpha_{m-1}+j_{p_m}} X_{\alpha_m}, \\ X_{\alpha_1} \cdots X_{\alpha_{m-3}+j_{p_{m-2}}} X_{\alpha_{m-2}} X_{\alpha_{m-1}} X_{\alpha_{m-1}+1} \cdots X_{\alpha_m-1}).$

Corollary 2.2. Let \mathcal{K} be the subgraph of \mathcal{K}' having loops only in the vertices of K_m . The ideal of vertex covers of $I(\mathcal{K})$ has at most $m + 1$ monomial generators. For instance, if the loops lie on $v_{\alpha_2}, v_{\alpha_4}, v_{\alpha_5}, v_{\alpha_{m-3}}, v_{\alpha_{m-1}}$, it is:
 $I_c(\mathcal{K}) = (X_{\alpha_1} \cdots X_{\alpha_m}, X_1 \cdots X_{\alpha_1-1} X_{\alpha_2} \cdots X_{\alpha_m}, X_{\alpha_1} X_{\alpha_1+1} \cdots X_{\alpha_2-1} X_{\alpha_2} X_{\alpha_3} \cdots X_{\alpha_m}, \dots, X_{\alpha_1} \cdots X_{\alpha_{m-2}} X_{\alpha_{m-2}+1} \cdots X_{\alpha_{m-1}-1} X_{\alpha_{m-1}} X_{\alpha_m}, X_{\alpha_1} \cdots X_{\alpha_{m-1}} X_{\alpha_{m-1}+1} \cdots X_{\alpha_m-1})$.

Corollary 2.3. Let $\mathcal{H} \cup \{v_h, v_h\}$ be the connected graph on n vertices $v_1, \dots, v_m, \dots, v_n$, $\forall h = m + 1, \dots, n$. The ideal of vertex covers for the edge ideal of $\mathcal{H} \cup \{v_h, v_h\}$ is generated by the m monomials $X_2 X_3 \cdots X_m X_{m+1} \cdots X_n$, $X_1 X_3 \cdots X_m X_{m+1} \cdots X_n$, \dots , $X_1 X_2 \cdots X_{m-1} X_{m+1} \cdots X_n$.

Example 2.2. Let \mathcal{K}' be the connected graph with $V(\mathcal{K}') = \{v_1, v_2, v_3, \dots, v_{11}\}$ given by $K_3 \cup \text{star}_1(3) \cup \text{star}_2(3) \cup \text{star}_3(2) \cup \{X_3, X_3\} \cup \{X_5, X_5\} \cup \{X_7, X_7\} \cup \{X_9, X_9\}$. The ideal of vertex covers is
 $I_c(\mathcal{K}') = (X_1 X_2 X_3 X_5 X_7 X_9, X_2 X_3 X_4 X_5 X_6 X_7 X_9, X_1 X_3 X_5 X_7 X_8 X_9)$.

3 Ideals of vertex covers of linear type

Let R be a noetherian ring and let $I = (f_1, \dots, f_s)$ be an ideal of R . The Rees algebra $\mathfrak{R}(I)$ of I is defined to be the R -graded algebra $\bigoplus_{i \geq 0} I^i$. It can be identified with the R -subalgebra of $R[t]$ generated by It , where t is an indeterminate on R . Let us consider the epimorphism of graded R -algebras $\varphi : R[T_1, \dots, T_s] \rightarrow \mathfrak{R}(I) = R[f_1 t, \dots, f_s t]$ defined by $\varphi(T_i) = f_i t$, $i = 1, \dots, s$.

The ideal $N = \ker \varphi$ of $R[T_1, \dots, T_s]$ is R -homogeneous and we denote N_i the R -homogeneous component of degree i of N . The elements of N_1 are called linear relations. If $A = (a_{ij})$, $i = 1, \dots, r$, $j = 1, \dots, s$ is the relation matrix of I , then $g_i = \sum_{j=1}^s a_{ij} T_j$, $i = 1, \dots, r$, belong to N and $R[T_1, \dots, T_s]/J$, with $J = (g_1, \dots, g_r)$, is isomorphic to the symmetric algebra $\text{Sym}_R(I)$ of I . The generators g_i of J are all linear in the variables T_j .

The natural map $\psi : \text{Sym}_R(I) \rightarrow \mathfrak{R}(I)$ is a surjective homomorphism of R -algebras. I is called of *linear type* if ψ is an isomorphism, that is $N = J$. Now, let K be a field, $R = K[X_1, \dots, X_n]$ be the polynomial ring, $I \subset R$ be a graded ideal whose generators f_1, \dots, f_s are all of the same degree. Let $S = R[T_1, \dots, T_s]$ be the polynomial ring over R in the variables T_1, \dots, T_s . Then we define a bigrading of S by setting $\deg(X_i) = (1, 0)$ for $i = 1, \dots, n$ and $\deg(T_j) = (0, 1)$ for $j = 1, \dots, s$. Consider the presentation $\varphi : R[T_1, \dots, T_s] \rightarrow \mathfrak{R}(I)$, $\varphi(T_i) = f_i t$, $i = 1, \dots, s$. If $I = (f_1, \dots, f_s) \subset R$ is a monomial ideal, for all $1 \leq i < j \leq s$ we set $f_{ij} = \frac{f_i}{\text{GCD}(f_i, f_j)}$ and $g_{ij} = f_{ij} T_j - f_{ji} T_i$, then J is generated by $\{g_{ij}\}_{1 \leq i < j \leq s}$.

Our aim is to investigate classes of monomial ideals arising from graphs for

which the linear relations g_{ij} form a system of generators for N , i.e. $N = J$. Consider the ideals of vertex covers $I_c(\mathcal{H})$ and $I_c(\mathcal{K}')$.

Theorem 3.1. *Let $R = K[X_1, \dots, X_m, \dots, X_n]$. $I_c(\mathcal{H})$ is of linear type.*

Proof. Let f_1, \dots, f_{m+1} be the minimal system of monomial generators of $I_c(\mathcal{H})$, where $f_1 = X_{\alpha_1} X_{\alpha_2} \cdots X_{\alpha_m}$, $f_2 = X_1 \cdots X_{\alpha_1-1} X_{\alpha_2} \cdots X_{\alpha_m}$, \dots , $f_m = X_{\alpha_1} \cdots X_{\alpha_{m-2}} X_{\alpha_{m-2}+1} \cdots X_{\alpha_{m-1}-1} X_{\alpha_m}$, $f_{m+1} = X_{\alpha_1} \cdots X_{\alpha_{m-1}} X_{\alpha_{m-1}+1} \cdots X_{\alpha_m-1}$ (Theorem 2.1).

We prove that the linear relations $g_{ij} = f_{ij}T_j - f_{ji}T_i$ form a Gröbner basis of N with respect to a monomial order \prec on the polynomial ring $R[T_1, \dots, T_{m+1}]$. Denote by H the ideal $(f_{ij}T_j : 1 \leq i < j \leq m+1)$. To show that g_{ij} are a Gröbner basis of N we suppose that the claim is false. Since the binomial relations are known to be a Gröbner basis of N , there exists a binomial $\underline{X}^a \underline{T}^\alpha - \underline{X}^b \underline{T}^\beta \in N$, where $\underline{X}^a = X_1^{a_1} \cdots X_n^{a_n}$, $\underline{X}^b = X_1^{b_1} \cdots X_n^{b_n}$, $\underline{T}^\alpha = T_1^{\alpha_1} \cdots T_{m+1}^{\alpha_{m+1}}$, $\underline{T}^\beta = T_1^{\beta_1} \cdots T_{m+1}^{\beta_{m+1}}$, and the initial monomial of $\underline{X}^a \underline{T}^\alpha - \underline{X}^b \underline{T}^\beta$ is not in H . More precisely, we assume that $\underline{T}^\alpha, \underline{T}^\beta$ have no common factors and that both $\underline{X}^a \underline{T}^\alpha$ and $\underline{X}^b \underline{T}^\beta$ are not in H .

Let i be the smallest index such that T_i appears in \underline{T}^α or in \underline{T}^β . Since $\underline{X}^a \underline{T}^\alpha - \underline{X}^b \underline{T}^\beta \in N$, then f_i divides $\underline{X}^b \varphi(\underline{T}^\beta)$, where $\varphi(T_i) = f_i T_i$. If $f_i | \underline{X}^b$, then let T_j be any of the variables of \underline{T}^β . One has $f_{ij}T_j | f_i T_j | \underline{X}^b \underline{T}^\beta$ for $i < j$. This is a contradiction by assumption (because $\underline{X}^b \underline{T}^\beta \notin H$).

Hence $f_i \nmid \underline{X}^b$. Let $X_{i_1} \prec \dots \prec X_{i_s}$ be a total term order on the variables of f_i , and let $f_i = X_{i_1} \cdots X_{i_s}$. Let $i_{k_1}, \dots, i_{k_t} \in \{i_1, \dots, i_s\}$ be the indices such that $X_{i_{k_1}}, \dots, X_{i_{k_t}}$ don't divide \underline{X}^b and i_{k_1} be the minimum of the indices such that $X_{i_{k_1}}$ does not divide \underline{X}^b . Then $g_i = f_i / X_{i_{k_1}} \cdots X_{i_{k_t}}$ divides \underline{X}^b . Since $X_{i_{k_1}}$ divides $\underline{X}^b \varphi(\underline{T}^\beta)$ (because $f_i | \underline{X}^b \varphi(\underline{T}^\beta)$), then there exists j such that T_j appears in \underline{T}^β and $X_{i_{k_1}} | f_j$. By the structure of the generators f_1, \dots, f_{m+1} of $I_c(\mathcal{H})$ (see Theorem 2.1) if $X_{i_{k_1}} | f_i$ and $X_{i_{k_1}} | f_j$ with j such that T_j is in \underline{T}^β , then $f_{ij} | g_i$. Hence f_{ij} divides \underline{X}^b and, as a consequence, $f_{ij}T_j$ divides $\underline{X}^b \underline{T}^\beta$, that is a contradiction (because $\underline{X}^b \underline{T}^\beta \notin H$). It follows that $N = (g_{ij} : 1 \leq i < j \leq m+1)$, hence $I_c(\mathcal{H})$ is of linear type. \square

Theorem 3.2. *Let $R = K[X_1, \dots, X_m, \dots, X_n]$. $I_c(\mathcal{K}')$ is of linear type.*

Proof. Let f_1, \dots, f_ℓ for $\ell \leq m+1$ be the minimal system of monomial generators of $I_c(\mathcal{K}')$ described in Theorem 2.2.

We prove that the linear relations $g_{ij} = f_{ij}T_j - f_{ji}T_i$, $1 \leq i < j \leq \ell$, form a Gröbner basis of N . We suppose that the claim is false. Using the same notation of Theorem 3.1 there exists a binomial $\underline{X}^a \underline{T}^\alpha - \underline{X}^b \underline{T}^\beta \in N$ and $\text{in}_\prec(\underline{X}^a \underline{T}^\alpha - \underline{X}^b \underline{T}^\beta)$ is not in $H = (f_{ij}T_j : 1 \leq i < j \leq \ell, \ell \leq m+1)$.

Let i be the smallest index such that T_i appears in \underline{T}^α or in \underline{T}^β . Since $\underline{X}^a \underline{T}^\alpha - \underline{X}^b \underline{T}^\beta \in N$, then f_i divides $\underline{X}^b \varphi(\underline{T}^\beta)$, where $\varphi(T_i) = f_{it}$. If $f_i | \underline{X}^b$, then $f_{ij} T_j | f_i T_j | \underline{X}^b \underline{T}^\beta$, $i < j$ and T_j any of the variables of \underline{T}^β . This is a contradiction.

Hence $f_i \nmid \underline{X}^b$. Let $f_i = X_{i_1} \dots X_{i_s}$, $i_{k_1}, \dots, i_{k_t} \in \{i_1, \dots, i_s\}$ be the indices such that $X_{i_{k_1}}, \dots, X_{i_{k_t}}$ don't divide \underline{X}^b and i_{k_1} be the minimum of the indices such that $X_{i_{k_1}}$ does not divide \underline{X}^b . Set $g_i = f_i / X_{i_{k_1}} \dots X_{i_{k_t}}$. Since $X_{i_{k_1}}$ divides $\underline{X}^b \varphi(\underline{T}^\beta)$, then there exists j such that T_j appears in \underline{T}^β and $X_{i_{k_1}} | f_j$. By the structure of the monomials f_1, \dots, f_ℓ (Theorem 2.2) if $X_{i_{k_1}} | f_i$ and $X_{i_{k_1}} | f_j$ with j such that T_j is in \underline{T}^β , then $f_{ij} | g_i$. Hence $f_{ij} | \underline{X}^b$ and, as a consequence, $f_{ij} T_j | \underline{X}^b \underline{T}^\beta$, that is a contradiction. Hence $N = (g_{ij} : 1 \leq i < j \leq \ell, \ell \leq m+1)$. The thesis follows. \square

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